Quantum phase transitions in an exactly solvable quantum-spin biaxial model with multiple spin interactions

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A distinct integrable quantum-spin model is proposed. The model has a biaxial magnetic anisotropy of alternating coupling between spins together with multiple spin interactions. Our model gives the possibility to exactly find thermodynamic characteristics of the considered spin chain. The ground state of the model can reveal spontaneous values of the total magnetic and antiferromagnetic moments caused by multiple spin couplings. Also, in the ground state, depending on the strength of multiple spin couplings, our model manifests several quantum critical points, some of which are governed by the external magnetic field.

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Integrable models of quantum physics of magnetism are unfortunately rare. Such models, however, are very important for theorists because they permit to compare the results of standard for theoretical physics of real systems perturbative approaches with exact ones. For example, the seminal Ising model served as a basis for many powerful methods of modern physics, such as the scaling, renormalization group, etc. Quantum integrable models in one space dimension (1D) are developed relatively more, compared to the two-dimensional (2D) or three-dimensional (3D) ones (in fact there exist only few examples of quantum integrable models in 2D and 3D), due to the relative simplicity of their study. On the other hand, according to the Mermin-Wagner theorem, any nonzero temperature in 1D (and 2D) destroys long-range magnetic ordering. That theorem reveals, in fact, the enhancement of quantum and thermal fluctuations in low-dimensional systems due to peculiarities in their densities of states.

On the other hand, the interest in quantum-spin systems, where spin-spin interactions along one or two space directions are much stronger than couplings along other directions, has considerably grown during the last decade. Such interest to low-dimensional quantum-spin systems is motivated, first of all, by the progress in the preparation of real substances with well-defined one-dimensional subsystems. On the other hand, modern technologies permit us to compose artificial one-dimensional quantum systems, such as quantum wires and rings, properties of which are created to be similar to theoretically known models. Devices based on such especially prepared quantum 1D spin systems are very promising in modern nanotechnology, in the development of spintronics, or even in quantum computation.

1D quantum-spin systems often manifest quantum phase transitions, i.e., those which take place in the ground state, and which are governed by other than the temperature parameters, such as an external magnetic field, external or internal (caused by chemical substitutions) pressure, etc.

In the past most of exactly solvable quantum-spin models were related to the class of models with only spin-spin interactions between nearest-neighbor (NN) spins.[1] During the past few years more attention of physicists was paid to theoretical studies of quantum-spin models with not only nearest-neighbor spin-spin interactions but also with next-nearest-neighbor ones, multiple spin-exchange models (e.g., a ring exchange), etc.[2–9] Such models sometimes appear to be closer to the real situation in quasi-one-dimensional magnets, comparing to the ones with only nearest-neighbor couplings between only pairs of spins. For example, such additional interactions are often present in oxides of transition metals, where the direct exchange between magnetic ions is complimented by the superexchange between magnetic ions via nonmagnetic ones. Terms involving the product of three and four spin operators or more were only recently recognized to be important for the theoretical description of many physical systems despite the fact that multiple spin-exchange models were introduced by Thouless[10] already in 1965. For example, multiple spin exchanges were used for the description of the magnetic properties of solid ^{3}He.[11] Later similar models were used to study some cuprates[12] and spin ladders.[13] Often quantum-spin models with antiferromagnetic interactions and multiple spin interactions manifest the spin frustration, i.e., the lowest in energy state is highly degenerate.[14] Last but not the least, such models often reveal quantum phase transitions. For instance, many transition-metal compounds, such as copper oxides, are believed to manifest features, characteristic for quantum phase transitions. However, it is known that for many quantum-spin models the standard quasiclassical theoretical description, based on the quantization of small deviations of classical vectors of magnetization of magnetic sublattices yield incorrect results, especially in the vicinity of quantum critical points.[15] This is why, quantum exactly solvable spin models with multiple spin-exchange interactions, even being rather formal, and, sometimes, nonrealistic, are of great importance: They provide the possibility to check approximate theoretical methods, used for the description of more realistic physical models of quantum-spin systems with spin frustration.

In this paper we propose a distinct integrable model of quantum spins with nearest-neighbor interactions and multiple spin exchanges. The aim of this paper is to study a model that, on the one hand, contains multiple spin interac-

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tions, which usually produce incommensurate magnetic structures.[6] Second, the proposed model consists alternating exchange interactions between nearest neighbors, which can be the reason for the spin gap for low-lying excitations.[1] Finally, the model has the biaxial magnetic anisotropy, which is believed to be the key property of transition-metal compounds with strong spin-orbit coupling.[16] The 2D counterpart of the model can be related to the plaquette model of p_1+ip_2 superconducting arrays.[17] On the other hand, the model is relatively simple because the Hamiltonian of the model can be exactly transformed to the one of the free fermion lattice gas, and hence, most of thermodynamic characteristics can be calculated explicitly.

The Hamiltonian of our exactly solvable quantum-spin model with alternating nearest-neighbor couplings and three-spin interactions, which permits exact solution has the form

$$\begin{aligned}\mathcal{H} = &-H\sum_n (\mu_1 S^z_{n,1} + \mu_2 S^z_{n,2}) - \sum_n (J_{1x} S^x_{n,1} S^x_{n,2} + J_{1y} S^y_{n,1} S^y_{n,2}) \\ &- \sum_n (J_{2x} S^x_{n,2} S^x_{n+1,1} + J_{2y} S^y_{n,2} S^y_{n+1,1}) \\ &- J_{13}\sum_n (S^x_{n,1} S^z_{n,2} S^x_{n+1,1} + S^y_{n,1} S^z_{n,2} S^y_{n+1,1}) \\ &- J_{23}\sum_n (S^x_{n,2} S^z_{n+1,1} S^x_{n+1,2} + S^y_{n,2} S^z_{n+1,1} S^y_{n+1,2}),\end{aligned} \tag{1}$$

where $S^{x,y,z}_{n,1,2}$ are the operators of the spin-$\frac{1}{2}$ projections of the spin in the nth cell, which belongs to the sublattice one or two, $\mu_{1,2}$ are the effective magnetons of the sublattices, H is the external magnetic field directed along z axis, $J_{1,2,x,y}$ are the alternating exchange coupling constants between nearest-neighbor spins in the cell and between cells, and $J_{1,23}$ are the alternating coupling constants for three-spin interactions. Notice that the model reveals the biaxial magnetic anisotropy, i.e., the exchange interactions (in the spin subspace) along the x, y, and z directions are different. In the case $J_{1,2x}=J_{1,2y}$, i.e., in the case of only uniaxial magnetic anisotropy, the model contains, as a special case, the model studied in Ref. 18. On the other hand, the special case $J_{13}=J_{23}=0$ of the model is known for many years.[19] Finally, in the absence of the magnetic field, $H=0$, and three-spin couplings, $J_{13}=J_{23}=0$, the model can be related to the so-called 1D quantum compass model (in the special case $J_{1x}=\alpha$, $J_{1y}=1-\alpha$, and $J_{2x}=0$ $J_{2y}=1$).[20] Terms, in the Hamiltonian, which describe three-spin couplings, obviously violate time reversal and parity symmetries of the system separately, but the combination of both symmetries is preserved.

After the Jordan-Wigner transformation[1,15]

$$S^z_{n,1,2} = \frac{1}{2}\sigma_{n,1,2} = \frac{1}{2} - a^\dagger_{n,1,2} a_{n,1,2},$$

$$S^+_{n,1} \equiv S^x_{n,1} + iS^y_{n,1} = \prod_{m<n} \sigma_{m,1}\sigma_{m,2} a_{n,1},$$

$$S^-_{n,1} \equiv S^x_{n,1} - iS^y_{n,1} = a^\dagger_{n,1} \prod_{m<n} \sigma_{m,1}\sigma_{m,2},$$

$$S^+_{n,2} = \prod_{m<n} \sigma_{m,1}\sigma_{m,2}\sigma_{n,1} a_{n,2},$$

$$S^-_{n,2} = a^\dagger_{n,2} \prod_{m<n} \sigma_{m,1}\sigma_{m,2}\sigma_{n,1}, \tag{2}$$

where $a^\dagger_{n,1,2}$ and $a_{n,1,2}$ are creation and destruction operators, which satisfy fermionic anticommutation relations and after the Fourier transform

$$a_{n,1,2} = N^{-1/2}\sum_k a_{k,1,2} \exp(ikn) \tag{3}$$

and similar for $a^\dagger_{n,1,2}$, where N is the number of cells, Hamiltonian (1) gets the form

$$\begin{aligned}\mathcal{H} = \sum_k \Bigg[&\left(\mu_1 H - \frac{J_{13}}{2}\cos k\right) a^\dagger_{k,1} a_{k,1} \\ &+ \left(\mu_2 H - \frac{J_{23}}{2}\cos k\right) a^\dagger_{k,2} a_{k,2} - \frac{1}{2}(J^+_1 + J^+_2 e^{-ik}) a^\dagger_{k,1} a_{k,2} \\ &- \frac{1}{2}(J^+_1 + J^+_2 e^{ik}) a^\dagger_{k,2} a_{k,1} - \frac{1}{2}(J^-_1 - J^-_2 e^{-ik}) a^\dagger_{k,1} a^\dagger_{k,2} \\ &- \frac{1}{2}(J^-_1 - J^-_2 e^{ik}) a_{k,2} a_{k,1} \Bigg] - \frac{\mu_1+\mu_2}{2} NH,\end{aligned} \tag{4}$$

where $J^\pm_{1,2} = (1/2)(J_{1,2x} \pm J_{1,2y})$. With the help of a unitary transformation this Hamiltonian can be diagonalized as

$$\mathcal{H} = \sum_k \sum_{j=1}^{2} \varepsilon_{k,j}\left(b^\dagger_{k,j} b_{k,j} - \frac{1}{2}\right), \tag{5}$$

where

$$\varepsilon^2_{k,1,2} = F_k \pm \sqrt{F^2_k - G_k} \tag{6}$$

and

$$\begin{aligned}F_k = \frac{1}{8}[&J^2_{1x} + J^2_{1y} + J^2_{2x} + J^2_{2y} + 2(J_{1x}J_{2y} + J_{1y}J_{2x})\cos k \\ &+ (J^2_{13} + J^2_{23})\cos^2 k + 4(\mu^2_1 + \mu^2_2)H^2 \\ &- 4(\mu_1 J_{13} + \mu_2 J_{23})H\cos k],\end{aligned}$$

$$\begin{aligned}G_k = \Bigg\{ &\mu_1\mu_2 H^2 - \frac{1}{2}(\mu_1 J_{23} + \mu_2 J_{13})H\cos k \\ &+ \frac{1}{4}[J_{13}J_{23}\cos^2 k - J_{1x}J_{1y} - J_{2x}J_{2y} \\ &- (J_{1x}J_{2x} + J_{1y}J_{2y})\cos k]\Bigg\}^2 + \frac{1}{16}(J_{1x}J_{2x} - J_{1y}J_{2y})^2 \sin^2 k.\end{aligned} \tag{7}$$

Using the standard particle-hole transformation one can get only positive eigenvalues of Hamiltonian (6). One can check that in the case $J_{1,23}=0$ the spectrum coincides with the one in Ref. 19, while for $J_{1,2x}=J_{1,2y}$ it reproduces the spectrum from Ref. 18. The energies of eigenstates of the first branch are non-negative for all parameters of the model. The ener-

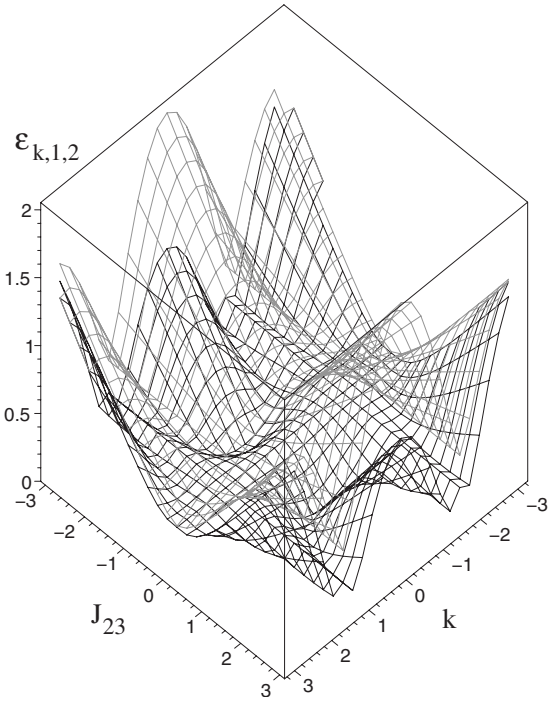

FIG. 1. Dispersion relations for the exactly solvable quantum compass model at zero magnetic field for the upper branch (gray) and lower branch (black) of eigenstates as functions of the parameter of three-spin interactions J_{23} for $J_{13}=J_{23}$.

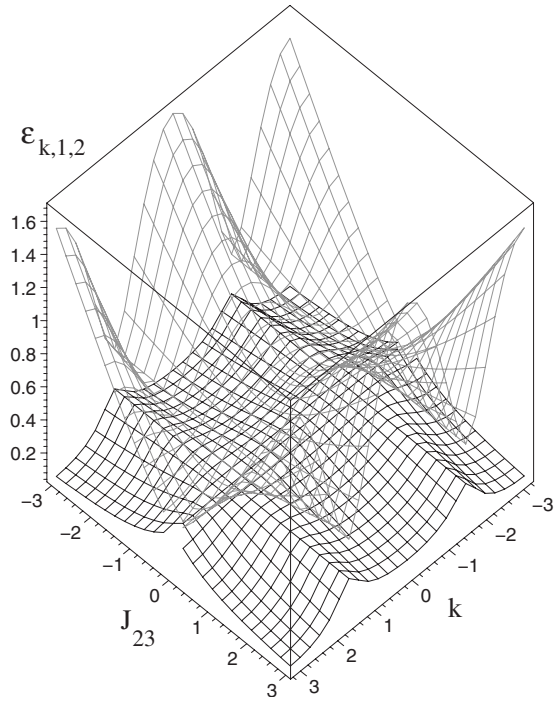

FIG. 2. The same as in Fig. 1, but for $J_{13}=0$.

gies of eigenstates belonging to the second branch can be equal to zero depending on the values of the parameters of the model. Figures 1 and 2 represent the zero-field dispersion relations for both branches as functions of three-spin interactions for the homogeneous three-spin couplings and for the alternating three-spin couplings, respectively, for the quantum compass model with $\alpha=0.4$

Despite some parameter-dependent features, the behavior of our model is similar for the quantum compass case with very strong biaxial magnetic anisotropy and the case with small biaxial anisotropy.

It is simple to obtain thermodynamic characteristics of our model at nonzero temperatures. The free energy of the quantum-spin chain is equal to

$$F = -T\sum_{k}\sum_{j=1}^{2}\ln\left(2\cosh\frac{\varepsilon_{k,j}}{2T}\right). \tag{8}$$

Obviously, the z projection of the average magnetization of the system is

$$M^z = \frac{1}{2}\sum_{k}\sum_{j=1}^{2}\frac{\partial\varepsilon_{k,j}}{\partial H}\tanh\left(\frac{\varepsilon_{k,j}}{2T}\right). \tag{9}$$

From this formula it is easy to show that M^z is zero for $H=0$ for any nonzero temperature, in accordance with the Mermin-Wagner theorem. The low-temperature behavior of the magnetic susceptibility,

$$\chi = \frac{1}{2}\sum_{k}\sum_{j=1}^{2}\left\{\frac{\partial^2\varepsilon_{k,j}}{\partial H^2}\tanh\left(\frac{\varepsilon_{k,j}}{2T}\right) + \left(\frac{\partial\varepsilon_{k,j}}{\partial H}\right)^2\left[2T\cosh\left(\frac{\varepsilon_{k,j}}{2T}\right)\right]^{-2}\right\}^{-1}, \tag{10}$$

and the specific heat,

$$c = \sum_{k}\sum_{j=1}^{2}\frac{\varepsilon_{k,j}^2}{4T^2\cosh^2(\varepsilon_{k,j}/2T)}, \tag{11}$$

depends on the values of coupling constants $J_{1,2,x,y}$, $J_{13,23}$, the effective magnetons, $\mu_{1,2}$, and the value of the external magnetic field H, see below. One can check that there is no ordering and, therefore, none of thermodynamic characteristics of the considered system has peculiarities at any nonzero temperature. On the other hand, as it will be shown below, in the ground state spontaneous magnetic ordering can take place. In the cases, where elementary excitations of the model are gapped, the low-temperature magnetic susceptibility and the specific heat reveal exponential in T dependencies in the absence of spontaneous magnetization. If the model reveals the spontaneous magnetic moment, the magnetic susceptibility at low temperatures is divergent. On the other hand, for gapless situation of low-energy states of the model, the magnetic susceptibility is finite at low temperatures for the absence of spontaneous magnetic ordering at $T=0$, while the specific heat is linear in T. At the critical lines of quantum phase transitions (see below) our model manifests either square root or logarithmic in T and magnetic field behaviors of the specific heat and the magnetic susceptibility. In the case, where interaction constants are very different from each other (or, to be more precise, when two branches of

eigenstates are characterized by very different energy scales), the specific heat and the magnetic susceptibility can reveal two maxima temperature dependencies.

The most important properties of the one-dimensional spin system are manifested in the ground state. The ground-state energy of our model can be written as

$$E_0 = -\frac{1}{\sqrt{2}}\sum_k \sqrt{F_k + \sqrt{G_k}}. \tag{12}$$

Then the z projections of each total spin moment of a cell in the ground state can be written as

$$S^z_{1,2} \equiv \frac{\partial E_0}{\partial \mu_{1,2} H} = \frac{1}{4\sqrt{2}}\sum_k \frac{2\sqrt{G_k}x_{1,2,k} + y_{1,2,k}}{\sqrt{G_k}\sqrt{F_k + \sqrt{G_k}}}, \tag{13}$$

where

$$x_{1,2,k} = \mu_{1,2}H - \frac{1}{2}J_{1,23}\cos k,$$

$$y_{1,2,k} = \Big\{4\mu_1\mu_2 H^2 - (\mu_1 J_{23} + \mu_2 J_{13})H\cos k + \frac{1}{2}[J_{13}J_{23}\cos^2 k - J_{1x}J_{1y} - J_{2x}J_{2y} - (J_{1x}J_{2x} + J_{1y}J_{2y})\cos k]\Big\} \times \left(\mu_{2,1}H - \frac{1}{2}J_{2,13}\cos k\right). \tag{14}$$

The sum of the z projections of spin moments can be considered as the ground-state vector of magnetism or magnetization of the model, $M^z = \mu_1 S^z_1 + \mu_2 S^z_2$, while the difference describes the vector of antiferromagnetism,[19] $L^z = \mu_1 S^z - \mu_2 S^z_2$, or the staggered magnetization of the model. From these expressions one immediately sees that in the ground state the model can have nonzero spontaneous magnetic and antiferromagnetic moments (i.e., magnetic ordering for $H=0$), caused by nonzero three-spin interactions. We would like to turn our attention to the fact that the signs of $J_{1,23}$ do matter. Namely, depending on their signs, the spontaneous magnetization of the model in the ground state can be positive or negative, with respect to the direction of the magnetic field. It is different from the behaviors of other known exactly solvable spin chain models. The reason for the onset of the spontaneous magnetic and antiferromagnetic moments in the ground state of our model is related to the violation of the time-reversal symmetry by three-spin coupling terms.

It is interesting to notice that the equality $G_k=0$ means that $\varepsilon_{k,2}=0$. As it is shown below, namely the condition $G_k=0$ determines the features in the behavior of all ground-state characteristics of the spin chain. One can see, that $G_k=0$ either at $\sin k=0$ (i.e., for $k=0,\pi$), or, for any k, if $J_{1x}J_{2x} = J_{1y}J_{2y}$ (it turns out that this condition does not depend on the magnetic field and on the values of three-spin couplings).

Let us consider the first the case with $J_{1x}J_{2x} \neq J_{1y}J_{2y}$. Notice that the limiting case of the quantum compass model belongs to the situation. The first branch of eigenstates is ever positive, but the second one can reach zero only for two values of the quasimomenta ($k=0,\pi$). Then it is simple to show that the ground-state magnetization is a continuous function of the external magnetic field, except at $H=0$ for $\mu_1 J_{23} \neq \mu_2 J_{13}$ and $\mu_1 J_{13} \neq \mu_2 J_{23}$, see Eqs. (13) and (14). For the latter the spontaneous magnetization appears, and therefore, the ground-state magnetic susceptibility is divergent at $H=0$. The magnetic susceptibility for nonzero values of H can have peculiarities, proportional to $\ln|H-H_{c,i}|$ ($i=1,2,3,4$), if the external magnetic field becomes equal to one of following four values

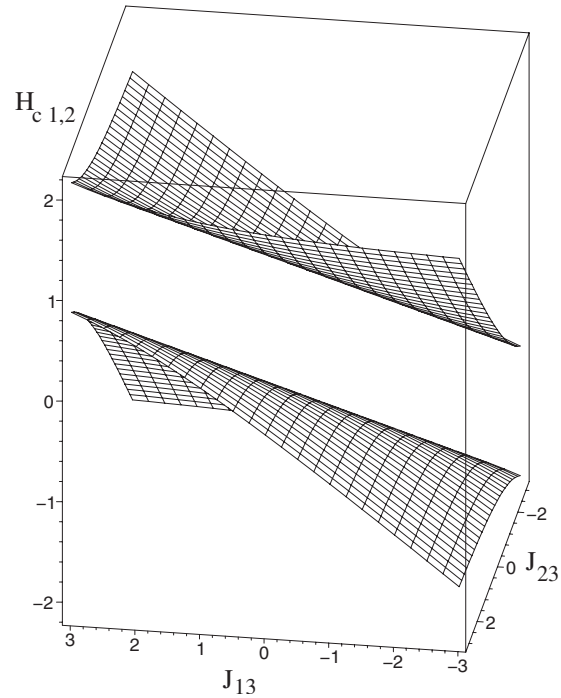

FIG. 3. Critical values of the magnetic fields $H_{c1,2}$ for the exactly solvable spin model as functions of parameters of three-spin interactions J_{13} and J_{23}. We used $\mu_1=1.01$, $\mu_2=0.99$, $J_{1x}=1$, $J_{1y}=1.5$, $J_{2x}=2$, and $J_{2y}=0.9$.

$$H_{c,1,2} = \frac{1}{4\mu_1\mu_2}[(\mu_1 J_{23} + \mu_2 J_{13})] \pm [(\mu_1 J_{23} - \mu_2 J_{13})^2 + \mu_1\mu_2(J_{1x} + J_{2y})(J_{1y} + J_{2x})]^{1/2},$$

$$H_{c,3,4} = \frac{1}{4\mu_1\mu_2}\{-(\mu_1 J_{23} + \mu_2 J_{13}) \pm [(\mu_1 J_{23} - \mu_2 J_{13})^2 + \mu_1\mu_2(J_{1x} - J_{2y})(J_{1y} - J_{2x})]^{1/2}\} \tag{15}$$

at which second-order quantum phase transitions can take place, see Figs. 3 and 4.

Such quantum critical points exist naturally only if those critical values of the field are real and non-negative. They are real if

$$|\mu_1\mu_2(J_{1x} \pm J_{2y})(J_{1y} \pm J_{2x})| \leq (\mu_1 J_{23} - \mu_2 J_{13})^2 \tag{16}$$

and the first two critical values are non-negative for $\mu_1\mu_2 > 0$ if

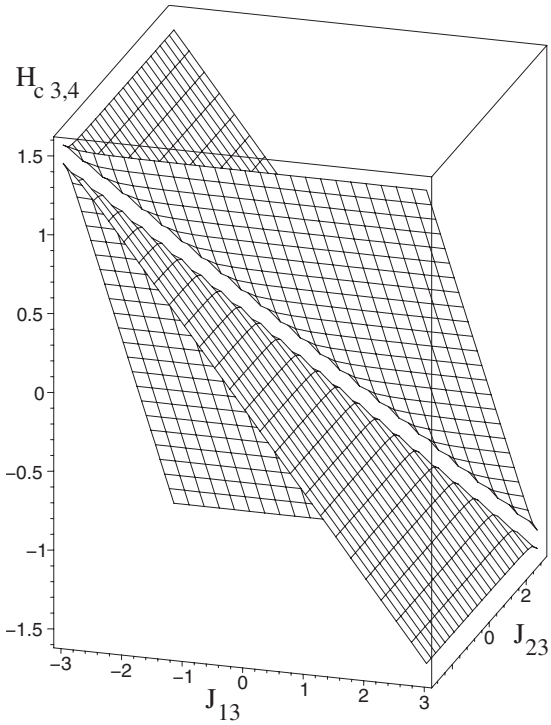

FIG. 4. Critical values of the magnetic fields $H_{c3,4}$ for the exactly solvable spin model as functions of parameters of three-spin interactions J_{13} and J_{23}. The set of parameters is the same as in Fig. 3.

$$(\mu_1 J_{23} + \mu_2 J_{13}) \geq [(\mu_1 J_{23} - \mu_2 J_{13})^2 + \mu_1\mu_2(J_{1x} + J_{2y})(J_{1y} + J_{2x})]^{1/2} > 0, \tag{17}$$

or the second two critical values are non-negative if

$$-(\mu_1 J_{23} + \mu_2 J_{13}) \geq [(\mu_1 J_{23} - \mu_2 J_{13})^2 + \mu_1\mu_2(J_{1x} - J_{2y})(J_{1y} - J_{2x})]^{1/2} > 0. \tag{18}$$

For $\mu_1\mu_2 < 0$ the non-negativity conditions are reversed. If the reality conditions are not satisfied, no quantum phase transitions governed by the external magnetic field take place in the system. If one of them is satisfied and the other is not, then only up to two quantum phase transitions can happen. If the conditions in Eq. (17) or Eq. (18) are not satisfied, then only one or two quantum phase transitions governed by the field take place. If one of the effective magnetons is zero (i.e., one of the ions, which form elementary cell is nonmagnetic), but the three-spin interaction constants are not, then the quantum phase transitions take place at the values of the magnetic field

$$H_{c5,6} = \pm \frac{J_{13}J_{23} + (J_{1x} \pm J_{2y})(J_{1y} \pm J_{2x})}{2\mu_{1,2}J_{2,13}}, \tag{19}$$

at which the magnetic susceptibility has logarithmic singularities. Naturally, only positive values of $H_{c5,6}$ matter. Finally, if both of effective magnetons are zero, then obviously, there are no quantum phase transitions governed by the magnetic field. It turns out that at nonzero temperatures thermodynamic characteristics of the model reveal logarithmic in T features at the critical values of the magnetic field.

Consider now the situation in which $J_{1x}J_{2x} = J_{1y}J_{2y}$. The energies of the eigenstates in this case can be written as

$$\varepsilon_{k,1,2} = \left[\left\{\frac{1}{4}\left[(\mu_1 + \mu_2)H - \frac{1}{2}(J_{13} + J_{23})\cos k\right]^2 + |B_k|^2\right\}^{1/2} \pm \left\{\frac{1}{4}\left[(\mu_1 - \mu_2)H - \frac{1}{2}(J_{13} - J_{23})\cos k\right]^2 + |A_k|^2\right]\right\}^{1/2}\right]^{1/2}, \tag{20}$$

where

$$A_k = \frac{1}{2}[J_1^+ + J_2^+ \exp(-ik)],$$

$$B_k = \frac{1}{2}[J_1^- - J_2^- \exp(-ik)]. \tag{21}$$

It is obvious that $\varepsilon_{k,1}$ is positive for any parameters of the model. On the other hand, for the lower branch for some ranges of the quasimomentum k and the external magnetic field H, the first term under the square root sign in Eq. (20) can be smaller than the second one. It implies that the eigenstates for lower branch can exist only for some ranges of k depending on the value of the external field H. The analysis of this situation is similar to the above (except the fact that one has to take into account nonzero Fermi seas, i.e., totally filled states with negative energies for some ranges of k depending on the value of the external field; the critical value of k is determined from the condition $\varepsilon_{2,k_c} = 0$). One can see that there exist four critical values of the magnetic field at which quantum phase transitions can take place, see Eqs. (15)–(19). The difference, comparing to the case with $J_{1x}J_{2x} \neq J_{1y}J_{2y}$, is in the more strong features of the magnetic susceptibility at critical fields $\sim 1/\sqrt{|H - H_{c,i}|}$ in the ground state, and therefore, in the square root peculiarities in T of thermodynamic characteristics of the model such as the magnetic susceptibility and specific heat, at critical values of the magnetic field.

Let us consider the homogeneous limiting case of our model $J_{1x} = J_{2x} = J_x$, $J_{1y} = J_{2y} = J_y$, $J_{13} = J_{23} = J_3$, and $\mu_1 = \mu_2 = \mu$. In this case the Hamiltonian can be written as

$$\mathcal{H} = \sum_k \varepsilon_k \left(b_k^\dagger b_k - \frac{1}{2}\right), \tag{22}$$

where

$$\varepsilon_k^2 = \left[\mu H - \frac{1}{2}(J_3 \cos(2k)) + (J_x + J_y)\cos(k)\right]^2 + \frac{1}{4}(J_x - J_y)^2 \sin^2(k). \tag{23}$$

One can see that the energy [Eq. (23)] is non-negative. It can be equal to zero only for $J_x = J_y$ or if $J_x \neq J_y$ for $k = 0, \pi$. In the later case there are two critical values of the magnetic field at

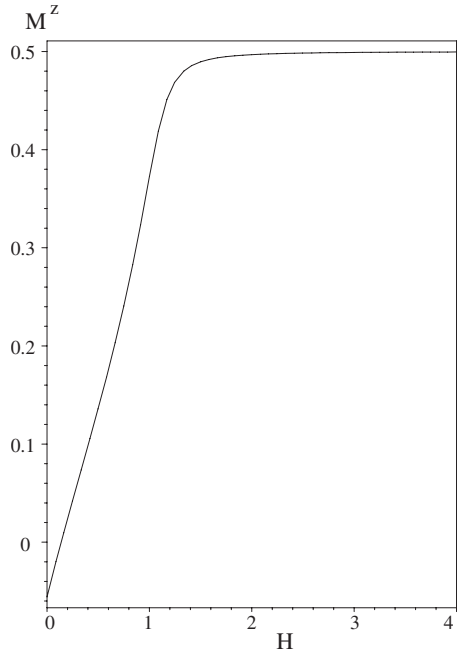

FIG. 5. The ground-state dependencies of the magnetization as a function of the magnetic field for the homogeneous limit of the exactly solvable model for $J_x=1$, $J_y=0.6$, and for $J_3=-2$.

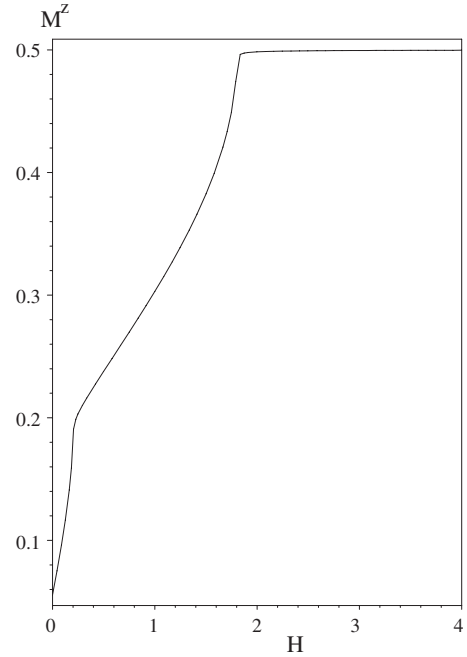

FIG. 6. The ground-state dependencies of the magnetization as a function of the magnetic field for the homogeneous limit of the exactly solvable model for $J_x=1$, $J_y=0.6$, and for $J_3=2$.

which quantum phase transitions can take place

$$H^h_{c,1,2} = (2\mu)^{-1}[J_3 \pm (J_x + J_y)]. \tag{24}$$

Obviously, quantum phase transitions take place if values of the critical field are non-negative, i.e., they take place for $\mu > 0$ if

$$J_3 \pm (J_x + J_y) \geq 0. \tag{25}$$

Hence, $J_3 = \pm(J_x+J_y)$ are the conditions of the quantum phase transition governed by the three-spin coupling. The ground-state magnetic susceptibility has logarithmic features $\sim \ln|h - H^h_{c1,2}|$ at critical values of the field.

Figures 5 and 6 show the ground-state behavior of the magnetization of our model for the homogeneous case as a function of the magnetic field. Figure 5 presents the behavior for $J_3 < -(J_x+J_y)$ and Fig. 6 demonstrates the magnetic field behavior for the region $J_3 > (J_x+J_y)$. One can see that for both regions there is a spontaneous magnetization, but its sign (with respect to the direction of the field) depends on the sign of three-spin interactions. Also, there are two quantum phase transitions for $J_3 > (J_x+J_y)$, while for $J_3 < -(J_x+J_y)$ the ground-state magnetization is a smooth function of H.

Figures 7 and 8 present the magnetic field behavior of the magnetization for $-(J_x+J_y) < J_3 < (J_x+J_y)$ for positive and negative values of J_3, respectively. One can see that in this region there is no spontaneous magnetization, and only one second-order quantum phase transition takes place.

On the other hand, if $J_x = J_y$, the eigenstates of the Hamiltonian can be negative for some ranges of k depending on the value of the magnetic field. Negative energies imply the nonzero Fermi sea, where eigenstates with negative energies are totally filled, and the ones with positive energies are empty. In that case quantum phase transitions yield square root singularities $\sim 1/\sqrt{|H - H^h_{c,1,2}|}$ of the magnetic susceptibility, cf. Refs. 7 and 8.

It is important to point out that the quasiclassical description of our model (when one replaces spin operators by classical vectors, and quantizing small deviations from the classical minimal energy state) does not reproduce exact results for the inhomogeneous (dimerized) situation. Namely, in the

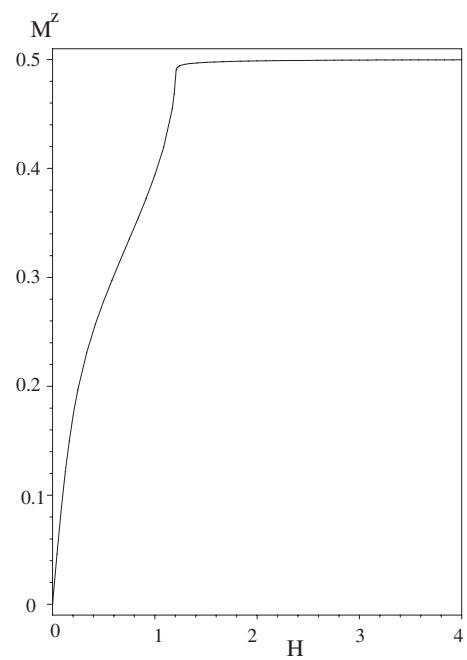

FIG. 7. The ground-state dependencies of the magnetization as a function of the magnetic field for the homogeneous limit of the exactly solvable model for $J_x=1$, $J_y=0.6$, and for $J_3=0.2$.

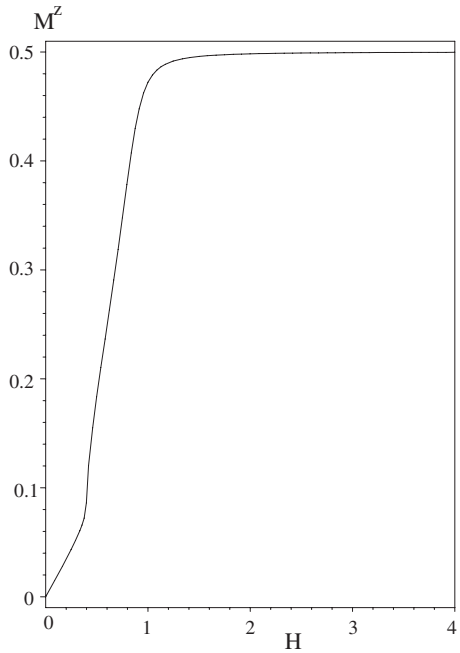

FIG. 8. The ground-state dependencies of the magnetization as a function of the magnetic field for the homogeneous limit of the exactly solvable model for $J_x=1$, $J_y=0.6$, and for $J_3=-0.8$.

classical description of the model without biaxial anisotropy one of the branches of eigenstates is obviously gapless, unlike the exact result.

In conclusion, motivated by recent experiments on quasi-1D quantum-spin systems and recent theories of quantum compass model, we propose the integrable model, in which exchange interactions between neighboring spins are accompanied by the multiple spin exchanges with the biaxial magnetic anisotropy. The model is simple (due to the exact mapping to the problem of the lattice free fermion gas), and therefore, permits to obtain exactly thermodynamic characteristics of the considered quantum-spin chain. The most important behavior of the model is in the ground state. Our model manifests a ferrimagneticlike ordering in the ground state. Depending on the signs of the parameters of three-spin couplings, the spontaneous magnetic moment of the system in the ground state can be positive or negative (with respect to the direction of the magnetic field). The system can undergo several second-order quantum phase transitions governed by the external magnetic field and the three-spin coupling strengths (the later can be caused by an external or internal pressure). Despite some artificial structures of our model, we expect that more realistic quantum biaxial spin systems with multiple exchange interactions and the alternation of the exchange between nearest-neighbor spins will show similar to our simple model behavior, i.e., our exact solution has generic features for this class of quantum systems.

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